Transforms -in the context of probability

Terminology

PDF: Probability Density Function

PMF: Probability Mass Function

Expectation:

$$E[X] = \int_{-\infty}^{\infty} x.f(x).dx$$

Drake and Bertsekas differ....

- The two texts diverge in their definitions of 'transforms' in the context of probability
 - Add into that the way I was taught as an Engineering Student!
- Drake goes into transforms more deeply and is closer to my understanding of s-transforms (Laplace)

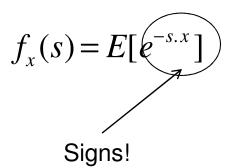
Bertsekas

$$M_X(s) = E[e^{sX}]$$

$$M(s) = \sum e^{s.x}.p_X(x)$$

$$M(s) = \int_{-\infty}^{\infty} e^{s.x} . f_X(x) . dx$$

Drake



The Drake definition is consistent with every other text I can find on the Laplace Transform! (and my engineering education)

Drake Ex.4.1 - Discrete

$$p_{x}(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 2\\ \frac{1}{6}, & \text{if } x = 3\\ \frac{1}{3}, & \text{if } x = 5 \end{cases}$$
 The PMF

$$M(s) = \frac{1}{2} \cdot e^{2 \cdot s} + \frac{1}{6} \cdot e^{3 \cdot s} + \frac{1}{3} \cdot e^{5 \cdot s}$$

Drake Ex.4.4-Linear function of RV

- This is an important result used later:
 - If X is a RV, and:

$$Y = a.X + b$$

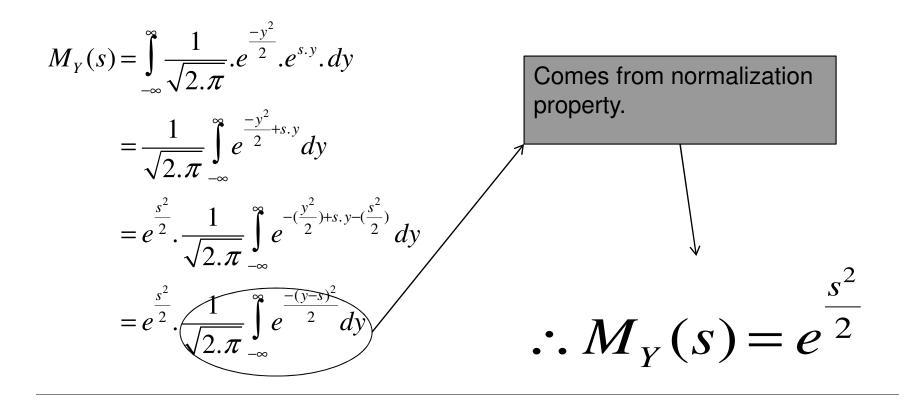
$$M_Y(s) = E\left[e^{s(a.X+b)}\right] = e^{sb}.E\left[e^{saX}\right] = e^{sb}M_X(sa)$$

Linear property of transforms

Drake Ex.4.5 - Continuous

Normal Distribution

$$f_Y(y) = \frac{1}{\sqrt{2.\pi}} e^{\frac{-y^2}{2}}$$



Transform of any Normal with non unity SD and Mean/Expectation

Using previous 2 results:

if
$$X = \sigma . Y + \mu$$

Then the transform of this new Normal distribution is:

$$M_X(s) = e^{s.\mu}.M_Y(s.\sigma) = e^{\left(\frac{\sigma^2.s^2}{2} + \mu.s\right)}$$

From Transforms to Moments

- 'Moment generating functions': Transforms
 - Different 'moments' are easily computed once we have the transform
 - Proof based on differentiating both sides of the original transform equation

$$M(s) = \int_{-\infty}^{\infty} e^{s.x} . f_X(x) . dx$$

$$\frac{d}{ds} M(s) = \frac{d}{ds} \int_{-\infty}^{\infty} e^{sx} . f_X(x) . dx$$

$$= \int_{-\infty}^{\infty} \frac{d}{ds} . e^{sx} . f_X(x) . dx$$

$$= \int_{-\infty}^{\infty} x . e^{sx} . f_X(x) . dx$$

Now consider case with s=0

$$\left. \frac{d}{ds} M(s) \right|_{s=0} = \int_{-\infty}^{\infty} x. f_X(x). dx = E[X]$$

It can be shown that this holds for the nth differential:

$$\left. \frac{d^n}{ds^n} M(s) \right|_{s=0} = \int_{-\infty}^{\infty} x^n \cdot f_X(x) \cdot dx = E[X^n]$$

This basically tells us that once we have the transform of the RV, then we can quickly compute the nth moment by taking the nth differential of that transform

Ex.4.6

• From Ex.4.1 $p_x(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 2\\ \frac{1}{6}, & \text{if } x = 3\\ \frac{1}{3}, & \text{if } x = 5 \end{cases}$ $M(s) = \frac{1}{2}.e^{2.s} + \frac{1}{6}.e^{3.s} + \frac{1}{3}.e^{5.s}$

$$M(s) = \frac{1}{2} \cdot e^{2 \cdot s} + \frac{1}{6} \cdot e^{3 \cdot s} + \frac{1}{3} \cdot e^{5 \cdot s}$$

$$E[X] = \frac{d}{ds} M(s) \Big|_{s=0}$$

$$= e^{2s} + \frac{1}{2} e^{3s} + \frac{5}{3} e^{5s} \Big|_{s=0}$$

$$= \frac{19}{6}$$

Transforms Part2 2/1/11

Inversion of Transforms

- Transforms are usually 'reversed' by recognizing a 'form' from a table.
- Ex.4.7

$$M(s) = \frac{1}{4} \cdot e^{-s} + \frac{1}{2} + \frac{1}{8} \cdot e^{4 \cdot s} + \frac{1}{8} \cdot e^{5 \cdot s}$$

This is of the form:

$$M(s) = \sum_{x} e^{s.x}.p_X(x)$$

And so we deduce that Px is a discrete random variable with:

$$P(X=-1) = \frac{1}{4}$$
 $P(X=0) = \frac{1}{2}$ $P(X=4) = \frac{1}{8}$ $P(X=5) = \frac{1}{8}$

Transform of a mix of two Dist.

- Ex.4.9
 - Combination of two exponential distributions

$$f_X(x) = \frac{2}{3}.6e^{-6x} + \frac{1}{3}.4e^{-4x}$$

$$M(s) = \int_0^\infty e^{s.x} \cdot \left(\frac{2}{3} \cdot 6e^{-6x} + \frac{1}{3} \cdot 4e^{-4x}\right) dx$$

$$M(s) = \frac{2}{3} \int_0^\infty e^{s.x} .6e^{-6x} dx + \frac{1}{3} \int_0^\infty e^{s.x} .4e^{-4x} dx$$

So the transform of the sum of two pdfs is just the sum of the transforms of the two pdfs

if
$$f_Y(y) = p_1 f_{x1}(y) + \dots + p_n f_{xn}(y)$$

then
$$M_Y(s) = p_1 M_{x1}(s) + \dots + p_n M_{xn}(s)$$

Sum of Independent RVs corresponds to multiplication of their transforms

- Note:
 - Multiplication in the transform domain is equivalent to convolution in the original variable or function
 - In this case the pdfs would be convolved with each other
- X and Y are independent RVs, W=X+Y

$$M_W(s) = E\left[e^{sW}\right] = E\left[e^{s(X+Y)}\right] = E\left[e^{sX}.e^{sY}\right]$$

If s is considered fixed:

$$M_W(s) = E\left[e^{sX}.e^{sY}\right] = E\left[e^{sX}\right].E\left[e^{sY}\right] = M_X(s).M_Y(s)$$

Generalizing: if

$$W = X_1 + \dots X_n,$$

then

$$M_W(s) = M_{X1}(s).M_{X2}(s)....M_{Xn}(s) = \prod M_{Xn}(s)$$

Sum of independent Normal RVs

X and Y are independent normal RVs with:

means:
$$\mu_x$$
 and μ_y
variances: $\sigma_x^2 \& \sigma_y^2$
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2.\sigma^2}}$$

$$M_X(s) = e^{\frac{\sigma_x^2 \cdot s^2}{2} + \mu_x^s}, \quad M_Y(s) = e^{\frac{\sigma_y^2 \cdot s^2}{2} + \mu_y^s}$$

$$M_W(s) = e^{\frac{\left(\sigma_x^2 + \sigma_y^2\right)s^2}{2} + \left(\mu_x + \mu_y\right)s}$$

The transform is the same as the transform of a single Normal RV -with the means and variances summed

Sums of Independent RVs

- If X & Y are independent RVs, then the distribution of their sum W=X+Y can be obtained by using the product of the transforms of the distributions of X and Y and then doing an inverse transform
- Alternatively it can be computed directly using convolution....

Discrete Case of Convolution

 X&Y are independent integer valued RVs with PMFs PX(x) and PY(y)

$$P_{W}(w) = P(X+Y)$$

$$= \sum_{(x,y):x+y=w} P(X=x \& Y=y) \qquad \text{for w being any integer}$$

$$= \sum_{x} P(X=x \& Y=(w-x))$$

$$= \sum_{x} p_{x}(x)p_{y}(w-x) \qquad \longleftarrow \qquad \boxed{\text{Convolution sum}}$$

Continuous case of convolution

 X & Y are independent continuous RVs with PDFs f(x) and f(y) respectively

$$F_{W}(w) = P(W \le w) \longleftarrow \text{Find the CDF expression for the summed RVs}$$

$$= P((X+Y) \le w)$$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{w-x} f_{X}(x) f_{Y}(y) dy. dx$$

$$= \int_{x=-\infty}^{\infty} f_{X}(x) \left[\int_{y=-\infty}^{w-x} f_{Y}(y). dy \right]. dx$$

$$= \int_{x=-\infty}^{\infty} f_{X}(x).F_{Y}(w-x) dx$$

Continued...

Continuous convolution cont.

 The PDFof W is then obtained by differentiating the expression for the CDF:

Ex 4.14

X&Y are independent and uniformly distributed in the interval [0,1] (and so the height of the pdf is 1). The PDF of W = X + Y:

$$f_{W}(w) = \int_{-\infty}^{\infty} f_{X}(x) f_{Y}(w - x) dx \qquad f_{X}(x) = \begin{cases} 1 \text{ for } 0 \le x \le 1 \\ 0 \text{ otherwize} \end{cases}$$

$$f_{Y}(y) = \begin{cases} 1 \text{ for } 0 \le (w - x) \le 1\\ 0 \text{ otherwize} \end{cases}$$

So from w=0 to w=2, small dx increments get summed to produce a triangular distribution

