The Power of Mediation in an Extended El Farol Game

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Abstract. A mediator implements a correlated equilibrium when it proposes a strategy to each player confidentially such that the mediator's proposal is the best interest for every player to follow. In this paper, we present a mediator that implements the best correlated equilibrium for an extended El Farol game with symmetric players. The extended El Farol game we consider incorporates both negative and positive network effects.

We study the degree to which this type of mediator can decrease the overall social cost. In particular, we give an exact characterization of *Mediation Value* (*MV*) and *Enforcement Value* (*EV*) for this game. *MV* is the ratio of the minimum social cost over all Nash equilibria to the minimum social cost over all mediators of this type, and *EV* is the ratio of the minimum social cost over all mediators of this type to the optimal social cost. This sort of exact characterization is uncommon for games with both kinds of network effects. An interesting outcome of our results is that both the *MV* and *EV* values can be unbounded for our game.

Keywords: Nash Equilibria, Correlated Equilibria, Mediators and Network Effects.

1 Introduction

When players act selfishly to minimize their own costs, the outcome with respect to the total social cost may be poor. The Price of Anarchy [1] measures the impact of selfishness on the social cost and is defined as the ratio of the worst social cost over all Nash equilibria to the optimal social cost. In a game, with a high Price of Anarchy, one way to reduce social cost is to find a mediator of expected social cost less than the social cost of any Nash equilibrium.

In the literature, there are several types of mediators [2,3,4,5,6,7,8,9,10,11]. In this paper, we consider only the type of mediator that implements a correlated equilibrium (CE) [12].

A mediator is a trusted external party that suggests a strategy to every player separately and privately so that each player has no gain to choose another strategy assuming that the other players conform to the mediator's suggestion.

The algorithm that the mediator uses is known to all players. However, the mediator's random bits are unknown. We assume that the players are symmetric in the sense that they have the same utility function and the probability the mediator suggests a strategy to some player is independent of the identity of that player.

Ashlagi et al. [13] define two metrics to measure the quality of a mediator: the mediation value (MV) and the enforcement value (EV). In our paper, we compute these values, adapted for games where players seek to minimize the social cost. The *Mediation Value* is defined as the ratio of the minimum social cost over all Nash equilibria to the minimum social cost over all mediators. The *Enforcement Value* is the ratio of the minimum social cost over all mediators to the optimal social cost.

A mediator is optimal when its expected social cost is minimum over all mediators. Thus, the *Mediation Value* measures the quality of the optimal mediator with respect to the best Nash equilibrium; and the *Enforcement Value* measures the quality of the optimal mediator with respect to the optimal social cost.

1.1   El Farol Game

First we describe the traditional El Farol game [14,15,16,17]. El Farol is a tapas bar in Santa Fe. Every Friday night, a population of people decide whether or not to go to the bar. If too many people go, they will all have a worse time than if they stayed home, since the bar will be too crowded. That is a negative network effect [18].

Now we provide an extension of the traditional El Farol game, where both negative and positive network effects [18] are considered. The positive network effect is that if too few people go, those that go will also have a worse time than if they stayed home.

Motivation. Our motivation for studying this problem comes from the following discussion in [18].

"It's important to keep in mind, of course, that many real situations in fact display both kinds of [positive and negative] externalities - some level of participation by others is good, but too much is bad. For example, the El Farol Bar might be most enjoyable if a reasonable crowd shows up, provided it does not exceed 60. Similarly, an on-line social media site with limited infrastructure might be most enjoyable if it has a reasonably large audience, but not so large that connecting to the Web site becomes very slow due to the congestion."

We note that our El Farol extension is one of the simplest, non-trivial problems for which a mediator can improve the social cost. Thus, it is useful for studying the power of a mediation.

Formal Definition of the Extended El Farol Game. We now formally define our game, which is non-atomic [19,20], in the sense that no individual player
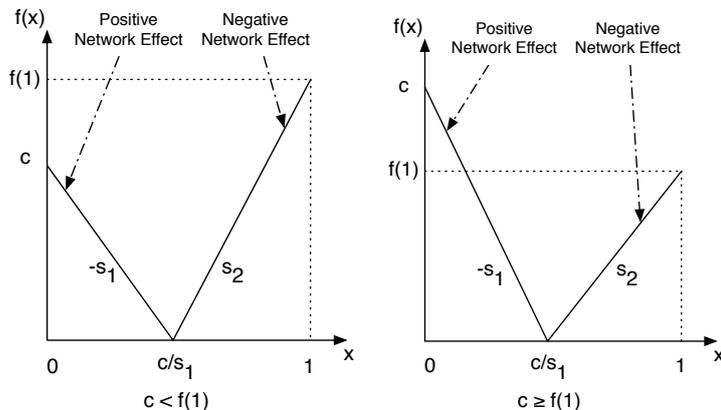
Fig. 1. The individual cost to go $f(x)$.

has significant influence on the outcome; moreover, the number of players is very large tending to infinity. The (c, s_1, s_2)-El Farol game has three parameters c, s_1 and s_2, where $0 < c < s_1$ and $s_2 > 0$. If x is the fraction of players to go, then the cost $f(x)$ for any player to go is as follows:

$$f(x) = \begin{cases} c - s_1 x & 0 \leq x \leq \frac{c}{s_1}, \\ s_2\left(x - \frac{c}{s_1}\right) & \frac{c}{s_1} \leq x \leq 1. \end{cases} \tag{1}$$

and the cost to stay is 1. The function $f(x)$ is illustrated in the two plots of Figure 1.

Our Contributions. The main contributions of our paper are threefold:

- We design an optimal mediator, which implements the best correlated equilibrium for an extension of the El Farol game with symmetric players. Notably, this extension incorporates both negative and positive network effects.
- We give an exact characterization of the *Mediation Value* (MV) and the *Enforcement Value* (EV) for our game.
- We show that both the MV and EV values can be unbounded for our game.

Paper Organization. In Section 2, we discuss the related work. Section 3 states the definitions and notations that we use in the El Farol game. Our results are given in Section 4, where we show our main theorem that characterizes the best correlated equilibrium, and we compute accordingly the *Mediation Value* and the *Enforcement Value*. Finally, Section 5 concludes the paper and discusses some open problems.

2   Related Work

2.1   Mediation Metrics

Christodoulou and Koutsoupias [21] analyze the price of anarchy and the price of stability for Nash and correlated equilibria in linear congestion games. A consequence of their results is that the EV for these games is at least 1.577 and at most 1.6, and the MV is at most 1.015.

Brandt et al. [22] compute the mediation value and the enforcement value in ranking games. In a ranking game, every outcome is a ranking of the players, and each player strictly prefers high ranks over lower ones [23]. They show that for the ranking games with $n > 2$ players, $EV = n - 1$. They also show that $MV = n - 1$ for $n > 3$ players, and for $n = 3$ players where at least one player has more than two actions.

The authors of [3] design a mediator that implements a correlated equilibrium for a virus inoculation game [24,25]. In this game, there are n players, each corresponding to a node in a square grid. Every player has either to inoculate itself (at a cost of 1) or to do nothing and risk infection, which costs $L > 1$. After each node decides to inoculate or not, one node in the grid selected uniformly at random is infected with a virus. Any node, v, that chooses not to inoculate becomes infected if there is a path from the randomly selected node to v that traverses only uninoculated nodes. A consequence of their result is that EV is $\Theta(1)$ and MV is $\Theta((n/L)^{1/3})$ for this game.

Jiang et al. [26] analyze the price of miscoordination (PoM) and the price of sequential commitment (PoSC) in security games, which are defined to be a certain subclass of Stackelberg games. A consequence of their results is that MV is unbounded in general security games and it is at least 4/3 and at most $\frac{e}{e-1} \approx 1.582$ in a certain subclass of security games.

We note that a poorly designed mediator can make the social cost worse than what is obtained from the Nash equilibria. Bradonjic et al. [27] describe the *Price of Mediation (PoM)* which is the ratio of the social cost of the worst correlated equilibrium to the social cost of the worst Nash equilibrium. They show that for a simple game with two players and two possible strategies, *PoM* can be as large as 2. Also, they show for games with more players or more strategies per player that *PoM* can be unbounded.

2.2   Finding and Simulating a Mediator

Papadimitriou and Roughgarden [28] develop polynomial time algorithms for finding correlated equilibria in a broad class of succinctly representable multi-player games. Unfortunately, their results do not extend to non-atomic games; moreover, they do not allow for direct computation of MV and EV, even when they can find the best correlated equilibrium.

Abraham et al. [29,30] describe a distributed algorithm that enables a group of players to simulate a mediator. This algorithm works robustly with up to linear size coalitions, and up to a constant fraction of adversarial players. The

result suggests that the concept of mediation can be useful even in the absence of a trusted external party.

2.3   Other Types of Mediators

In all equilibria above, the mediator does not act on behalf of the players. However, a more powerful type of mediators is described in [2,4,5,6,7,8,9,10,11], where a mediator can act on behalf of the players that give that right to it.

For multistage games, the notion of the correlated equilibrium is generalized to the communication equilibrium in [31,32]. In a communication equilibrium, the mediator implements a multistage correlated equilibrium; in addition, it communicates with the players privately to receive their reports at every stage and selects the recommended strategy to each player accordingly.

3   Definitions and Notations

Now we state the definitions and notations that we use in the El Farol game.

Definition 1. A configuration $C(x)$ *characterizes that a fraction of players, x, is being advised to go; and the remaining fraction of players, $(1-x)$, is being advised to stay.*

Definition 2. A configuration distribution $\mathcal{D}\{(C(x_1), p_1), .., (C(x_k), p_k)\}$ *is a probability distribution over $k \geq 2$ configurations, where $(C(x_i), p_i)$ represents that configuration $C(x_i)$ is selected with probability p_i, for $1 \leq i \leq k$. Note that $0 \leq x_i \leq 1$, $0 < p_i < 1$, $\sum_{i=1}^{k} p_i = 1$ and if $x_i = x_j$ then $i = j$ for $1 \leq i, j \leq k$.*

For any player i, let $\mathcal{E}_G^i$ be the event that player i is advised to go, and C_G^i be the cost for player i to go (when all other players conform to the advice). Also let $\mathcal{E}_S^i$ be the event that player i is advised to stay, and C_S^i be the cost for player i to stay. Since the players are symmetric, we will omit the index i.

A configuration distribution, $\mathcal{D}\{(C(x_1), p_1), .., (C(x_k), p_k)\}$, is a correlated equilibrium iff

$$\mathbf{E}\left[C_S | \mathcal{E}_G\right] \geq \mathbf{E}\left[C_G | \mathcal{E}_G\right],$$
$$\mathbf{E}\left[C_G | \mathcal{E}_S\right] \geq \mathbf{E}\left[C_S | \mathcal{E}_S\right].$$

Definition 3. A mediator *is a trusted external party that uses a configuration distribution to advise the players such that this configuration distribution is a correlated equilibrium. The set of configurations and the probability distribution are known to all players. The mediator selects a configuration according to the probability distribution. The advice the mediator sends to a particular player, based on the selected configuration, is known only to that player.*

Throughout the paper, we let n be the number of players.

4 Our Results

In our results, we assume that *the cost to stay* is 1; we justify this assumption at the end of this section. Our first results in Lemmas 1 and 2 are descriptions of the optimal social cost and the minimum social cost over all Nash equilibria for our extended El Farol game. We next state our main theorem which characterizes the best correlated equilibrium and determines the *Mediation Value* and *Enforcement Value*.

Lemma 1. *For any (c, s_1, s_2)-El Farol game, the optimal social cost is $(y^* f(y^*) + (1 - y^*))n$, where*

$$
y^* = \begin{cases}
\frac{1}{2}(\frac{c}{s_1} + \frac{1}{s_2}) & \text{if } \frac{c}{s_1} \le \frac{1}{2}(\frac{c}{s_1} + \frac{1}{s_2}) \le 1, \\
\frac{c}{s_1} & \text{if } \frac{1}{s_2} < \frac{c}{s_1}, \\
1 & \text{otherwise.}
\end{cases}
$$

Proof. By Equation (1), $f(x)$ has two cases. Let $f_1(x)$ be $f(x)$ for $x \in [0, \frac{c}{s_1}]$, and let $f_2(x)$ be $f(x)$ for $x \in [\frac{c}{s_1}, 1]$. Also let $h_1(x)$ be the social cost when $0 \le x \le \frac{c}{s_1}$, and let $h_2(x)$ be the social cost when $\frac{c}{s_1} \le x \le 1$. Thus, $h_1(x) = (x f_1(x) + (1 - x))n$ and $h_2(x) = (x f_2(x) + (1 - x))n$.

We know that $h_1(x)$ is minimized at $x = \frac{c}{s_1}$. In addition, we know that $h_2(x)$ is a quadratic function with respect to x, and thus it has one minimum over $x \in [\frac{c}{s_1}, 1]$ at $x = y^*$, where:

$$
y^* = \begin{cases}
\frac{1}{2}(\frac{c}{s_1} + \frac{1}{s_2}) & \text{if } \frac{c}{s_1} \le \frac{1}{2}(\frac{c}{s_1} + \frac{1}{s_2}) \le 1, \\
\frac{c}{s_1} & \text{if } \frac{1}{2}(\frac{c}{s_1} + \frac{1}{s_2}) < \frac{c}{s_1}, \\
1 & \text{otherwise.}
\end{cases}
$$

Let h^* be the optimal social cost. Then $h^* = min(h_1(\frac{c}{s_1}), h_2(y^*))$. Since $f_1(\frac{c}{s_1}) = f_2(\frac{c}{s_1})$, we have $h_1(\frac{c}{s_1}) = h_2(\frac{c}{s_1})$. Hence, $h^* = min(h_2(\frac{c}{s_1}), h_2(y^*))$. This implies that $h^* = h_2(y^*)$. □

Lemma 2. *For any (c, s_1, s_2)-El Farol game, if $f(1) \ge 1$, then the best Nash equilibrium is at which the cost to go in expectation is equal to the cost to stay; otherwise, the best Nash equilibrium is at which all players would rather go. The social cost of the best Nash equilibrium is $\min(n, f(1) \cdot n)$.*

Proof. There are two cases for $f(1)$ to determine the best Nash equilibrium.
Case 1: $f(1) \ge 1$. Let N_y be a Nash equilibrium with the minimum social cost over all Nash equilibria and with a y-fraction of players that go in expectation. If $f(y) > 1$, then at least one player of the y-fraction of players would rather stay. Also if $f(y) < 1$, then at least one player of the $(1 - y)$-fraction of players would rather go. Thus, we must have $f(y) = 1$. Assume that each player has a mixed strategy, where player i goes with probability y_i. Recall that N_y has a y-fraction of players that go in expectation. Thus, $y = \frac{1}{n} \sum_{i=1}^{n} y_i$. Then the social cost is $\sum_{i=1}^{n}(y_i f(y) + (1 - y_i))$, or equivalently, n.
Case 2: $f(1) < 1$. In this case, the best Nash equilibrium is at which all players would rather go, with a social cost of $f(1) \cdot n$.

Therefore, the social cost of the best Nash equilibrium is $min(n, f(1) \cdot n)$. □

Theorem 1. *For any (c, s_1, s_2)-El Farol game , if $c \leq 1$, then the best correlated equilibrium is the best Nash equilibrium; otherwise, the best correlated equilibrium is $\mathcal{D}\{(C(0), p), (C(x^*), 1-p)\}$, where $\lambda(c, s_1, s_2) = c(\frac{1}{s_1} + \frac{1}{s_2}) - \sqrt{\frac{c(\frac{1}{s_1} + \frac{1}{s_2})(c-1)}{s_2}}$,*

$$
x^* = \begin{cases} \lambda(c, s_1, s_2) & \text{if } \frac{c}{s_1} \leq \lambda(c, s_1, s_2) < 1, \\ \frac{c}{s_1} & \text{if } \lambda(c, s_1, s_2) < \frac{c}{s_1}, \\ 1 & \text{otherwise.} \end{cases}
$$

and $p = \frac{(1-x^*)(1-f(x^*))}{(1-x^*)(1-f(x^*))+c-1}$. Moreover,

1) *the expected social cost is $(p + (1-p)(x^* f(x^*) + (1-x^*)))n$,*
2) *the Mediation Value (MV) is $\frac{\min(f(1), 1)}{p + (1-p)(x^* f(x^*) + (1-x^*))}$ and*
3) *the Enforcement Value (EV) is $\frac{p + (1-p)(x^* f(x^*) + (1-x^*))}{y^* f(y^*) + (1-y^*)}$, where*

$$
y^* = \begin{cases} \frac{1}{2}(\frac{c}{s_1} + \frac{1}{s_2}) & \text{if } \frac{c}{s_1} \leq \frac{1}{2}(\frac{c}{s_1} + \frac{1}{s_2}) \leq 1, \\ \frac{c}{s_1} & \text{if } \frac{1}{s_2} < \frac{c}{s_1}, \\ 1 & \text{otherwise.} \end{cases}
$$

Due to the space constraints, the proof of this theorem is not given here.

The following corollary shows that for $c > 1$, if $\lambda(c, s_1, s_2) \geq 1$, then the best correlated equilibrium is the best Nash equilibrium, where all players would rather go.

Corollary 1. *For any (c, s_1, s_2)-El Farol game, if $c > 1$ and $\lambda(c, s_1, s_2) \geq 1$ then $MV = 1$.*

Proof. By Theorem 1, when $\lambda(c, s_1, s_2) \geq 1$, $x^* = 1$ and $p = 0$. Now we prove that if $\lambda(c, s_1, s_2) \geq 1$, then the best correlated equilibrium is the best Nash equilibrium of the case $f(1) < 1$ in Lemma 2. To do so, we prove that $\lambda(c, s_1, s_2) \geq 1 \Rightarrow f(1) < 1$.

Now assume by way of contradiction that $\lambda(c, s_1, s_2) \geq 1 \Rightarrow f(1) \geq 1$. Recall that $f(1) = s_2(1 - \frac{c}{s_1})$. Then $\lambda(c, s_1, s_2) \geq 1 \Rightarrow \frac{c}{s_1} + \frac{1}{s_2} \leq 1$, or equivalently, $\lambda(c, s_1, s_2) \geq 1 \Rightarrow \frac{c}{s_1} + \frac{1}{s_2} \leq \lambda(c, s_1, s_2)$. Also recall that $\lambda(c, s_1, s_2) = c(\frac{1}{s_1} + \frac{1}{s_2}) - \sqrt{\frac{c(\frac{1}{s_1} + \frac{1}{s_2})(c-1)}{s_2}}$. Thus, we have:

$$
\begin{aligned}
\lambda(c, s_1, s_2) \geq 1 \ &\Rightarrow \frac{c}{s_1} + \frac{1}{s_2} \leq c(\frac{1}{s_1} + \frac{1}{s_2}) - \sqrt{\frac{c(\frac{1}{s_1} + \frac{1}{s_2})(c-1)}{s_2}} \\
&\Rightarrow s_2 \cdot \frac{c}{s_1} \leq -1,
\end{aligned}
$$

which contradicts since s_1, s_2 and c are all positive. Therefore, for $c > 1$ and $\lambda(c, s_1, s_2) \geq 1$, MV must be equal to 1. $\qquad\square$

Now we show that MV and EV can be unbounded in the following corollaries.

Corollary 2. *For any* $(2 + \epsilon, \frac{2+\epsilon}{1-\epsilon}, \frac{1}{\epsilon})$-*El Farol game, as* $\epsilon \to 0$, $MV \to \infty$.

Proof. For any $(2 + \epsilon, \frac{2+\epsilon}{1-\epsilon}, \frac{1}{\epsilon})$-El Farol game, we have $f(1) = 1$. By Theorem 1, we obtain $x^* = 1 - \epsilon$, $f(x^*) = 0$ and $p = \frac{\epsilon}{1+2\epsilon}$ for $\epsilon \leq \frac{1}{2}(\sqrt{3} - 1)$. Thus we have

$$\lim_{\epsilon \to 0} MV = \lim_{\epsilon \to 0} \frac{\min\left(f(1), 1\right)}{\frac{\epsilon}{1+2\epsilon} + \epsilon(\frac{1+\epsilon}{1+2\epsilon})} = \infty.$$

$\square$

Corollary 3. *For any* $(1 + \epsilon, \frac{1+\epsilon}{1-\epsilon}, \frac{1}{\epsilon})$-*El Farol game, as* $\epsilon \to 0$, $EV \to \infty$.
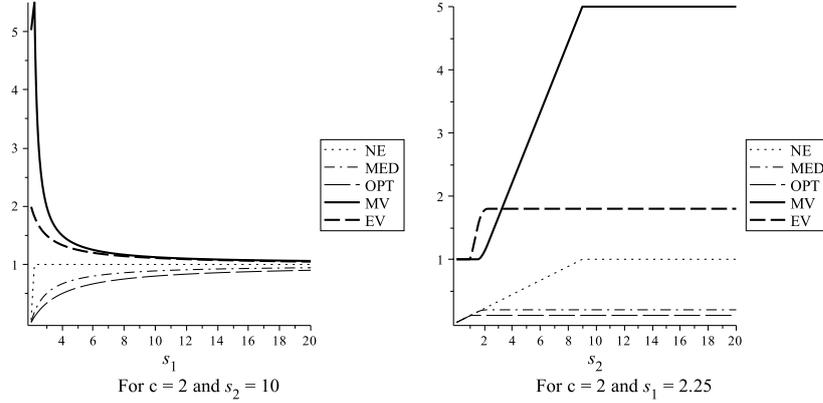
Proof. For any $(1 + \epsilon, \frac{1+\epsilon}{1-\epsilon}, \frac{1}{\epsilon})$-El Farol game, by Theorem 1, we obtain $x^* = 1 + \epsilon^2 - \epsilon\sqrt{1 + \epsilon^2}$ and $f(x^*) = 1 + \epsilon - \sqrt{1 - \epsilon^2}$. Then we have

$$p = \frac{(1 - (1 + \epsilon^2 - \epsilon\sqrt{1 + \epsilon^2}))(1 - (1 + \epsilon - \sqrt{1 - \epsilon^2}))}{(1 - (1 + \epsilon^2 - \epsilon\sqrt{1 + \epsilon^2}))(1 - (1 + \epsilon - \sqrt{1 - \epsilon^2})) + \epsilon}.$$
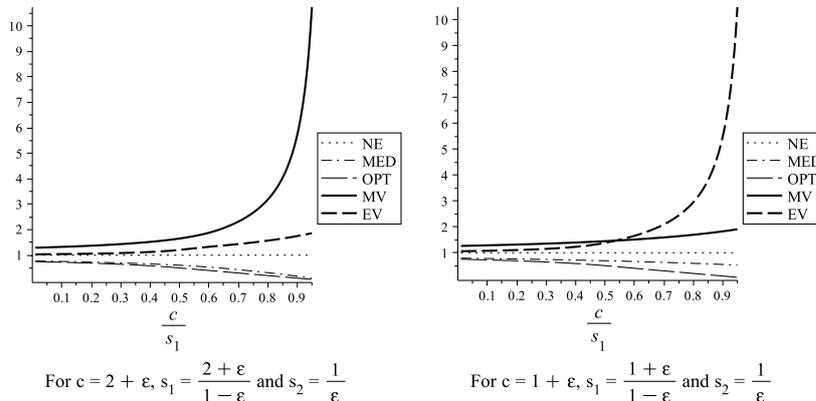
Also we have $y^* = 1 - \epsilon$ and $f(y^*) = 0$ for $\epsilon \leq \frac{1}{2}$. Thus we have

$$\lim_{\epsilon \to 0} EV = \lim_{\epsilon \to 0} \frac{p + (1 - p)(x^* f(x^*) + (1 - x^*))}{y^* f(y^*) + (1 - y^*)} = \infty.$$

$\square$

Fig. 2. NE, MED, OPT, MV and EV with respect to s_1 and s_2.

For c = 2 + ε, $s_1 = \dfrac{2+\varepsilon}{1-\varepsilon}$ and $s_2 = \dfrac{1}{\varepsilon}$ For c = 1 + ε, $s_1 = \dfrac{1+\varepsilon}{1-\varepsilon}$ and $s_2 = \dfrac{1}{\varepsilon}$

Fig. 3. NE, MED, OPT, MV and EV with respect to c/s_1.

Based on these results, we show in Figures 2 and 3 the social cost of the best Nash equilibrium (NE), the expected social cost of our optimal mediator (MED) and the optimal social cost (OPT), normalized by n, with respect to s_1, s_2 and c/s_1. Also we show the corresponding *Mediation Value* (MV) and *Enforcement Value* (EV).

In Figure 2, the left plot shows that for $c = 2$ and $s_2 = 10$, the values of NE, MED, OPT increase, each up to a certain point, when s_1 increases; however, the values of MV and EV decrease when s_1 increases. Moreover, MV reaches its peak at the point where the best Nash equilibrium starts to remain constant with respect to s_1. In the right plot, we set $c = 2$ and $s_1 = 2.25$; it shows that the values of NE, MED, OPT, MV and EV increase, each up to a certain point, when s_2 increases.

Figure 3 illustrates Corollaries 2 and 3, and it shows how fast MV and EV go to infinity with respect to c/s_1, where $c/s_1 = 1 - \epsilon$. The left plot shows that for any $(2 + \epsilon, \frac{2+\epsilon}{1-\epsilon}, \frac{1}{\epsilon})$-El Farol game, as $c/s_1 \to 1$ ($\epsilon \to 0$), $MV \to \infty$ and $EV \to 2$. In the right plot, for any $(1 + \epsilon, \frac{1+\epsilon}{1-\epsilon}, \frac{1}{\epsilon})$-El Farol game, as $c/s_1 \to 1$ ($\epsilon \to 0$), $EV \to \infty$ and $MV \to 2$.

Note that for any (c, s_1, s_2)-El Farol game, if $c/s_1 = 1$, then the best correlated equilibrium is at which all players would rather go with a social cost of 0, that is the best Nash equilibrium as well. Therefore, once c/s_1 is equal to 1, MV drops to 1.

The cost to stay assumption

Now we justify our assumption that the cost to stay is unity. Let (c', s'_1, s'_2, t')-El Farol game be a variant of (c, s_1, s_2)-El Farol game, where $0 < c' < s'_1$, $s' > 0$ and the cost to stay is $t' > 0$. If x is the fraction of players to go, then the cost

$f'(x)$ for any player to go is as follows:

$$f'(x) = \begin{cases} c' - s_1'x & 0 \leq x \leq \frac{c'}{s_1'}, \\ s_2'(x - \frac{c'}{s_1'}) & \frac{c'}{s_1'} \leq x \leq 1. \end{cases}$$

The following lemma shows that any (c', s_1', s_2', t')-El Farol game can be reduced to a (c, s_1, s_2)-El Farol game.

Lemma 3. *Any (c', s_1', s_2', t')-El Farol game can be reduced to a (c, s_1, s_2)-El Farol game that has the same* Mediation Value *and* Enforcement Value, *where $c = \frac{c'}{t'}$, $s_1 = \frac{s_1'}{t'}$ and $s_2 = \frac{s_2'}{t'}$.*

Proof. In a manner similar to Theorem (1), for any (c', s_1', s_2', t')-El Farol game, if $c > t'$, then the best correlated equilibrium is $\mathcal{D}\{(C(0), p'), (C(x'), 1 - p')\}$, where $\lambda'(c', s_1', s_2', t') = c'(\frac{1}{s_1'} + \frac{1}{s_2'}) - \sqrt{\frac{c'(\frac{1}{s_1'} + \frac{1}{s_2'})(c' - t')}{s_2'}}$;

$$x' = \begin{cases} \lambda'(c', s_1', s_2', t') & \text{if } \frac{c'}{s_1'} \leq \lambda'(c', s_1', s_2', t') < 1, \\ \frac{c'}{s_1'} & \text{if } \lambda'(c', s_1', s_2', t') < \frac{c'}{s_1'}, \\ 1 & \text{otherwise.} \end{cases}$$

and $p' = \frac{(1-x')(t'-f(x'))}{(1-x')(t'-f(x'))+c'-t'}$. Moreover,

1) the Mediation Value (MV') is $\frac{\min(f'(1), t')}{p't' + (1-p')(x'f(x') + (1-x')t')}$ and
2) the Enforcement Value (EV') is $\frac{p't' + (1-p')(x'f(x') + (1-x')t')}{y'f(y') + (1-y')t'}$, where

$$y' = \begin{cases} \frac{1}{2}(\frac{c'}{s_1'} + \frac{t'}{s_2'}) & \text{if } \frac{c'}{s_1'} \leq \frac{1}{2}(\frac{c'}{s_1'} + \frac{t'}{s_2'}) \leq 1, \\ \frac{c'}{s_1'} & \text{if } \frac{t'}{s_2'} < \frac{c'}{s_1'}, \\ 1 & \text{otherwise.} \end{cases}$$

Similarly, for $c \leq t'$, we have $MV' = 1$ and $EV' = \frac{\min(f'(1), t')}{y'f(y') + (1-y')t'}$.

For both cases, by Theorem 1, if we set $c = c'/t'$, $s_1 = s_1'/t'$ and $s_2 = s_2'/t'$, then we have $f'(1) = f(1) \cdot t'$; also we get $y' = y^*$ and $\lambda'(c', s_1', s_2', t') = \lambda(c, s_1, s_2)$. This implies that $f'(y') = f(y^*) \cdot t'$ and $x' = x^*$; which in turn $f'(x') = f(x^*) \cdot t'$ and $p' = p$. Thus, we obtain $MV' = MV$ and $EV' = EV$. □

5   Conclusion

We have extended the traditional El Farol game to have both negative and positive network effects. We have described an optimal mediator, and we have measured the *Mediation Value* and the *Enforcement Value* to completely characterize the benefit of our mediator with respect to the best Nash equilibrium and the optimal social cost.

Several open questions remain including the following: can we generalize our results for our game where the players choose among $k > 2$ actions? How many configurations are required to design an optimal mediator when there are $k > 2$ actions? Another problem is characterizing the MV and EV values for our game with the more powerful mediators in [2,4,5,6,7,8,9,10,11]. How much would these more powerful mediators reduce the social cost over our type of weaker mediator?

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A Appendix - Proof of Theorem 1

First of all, we call a mediator over k configurations when the configuration distribution this mediator uses has k configurations.

The proof has four main parts. The first part is *The Reduction of Mediators for $c > 1$*, where we prove that if $c > 1$, then for any optimal mediator over $k > 2$ configurations, there is a mediator over two configurations that has the same social cost. The second part is *The Reduction of Mediators for $c \leq 1$*, where we prove that if $c \leq 1$, then the best correlated equilibrium is the best Nash equilibrium. The third part is *An Optimal Mediator*, where we describe an optimal mediator for any arbitrary constants c, s_1 and s_2. Finally, the fourth part is *The Mediation Metrics*, where we measure the *Mediation Value* and the *Enforcement Value*.

Recall that x_i is the fraction of players that are advised to go in configuration $C(x_i)$ which is selected with probability p_i in a configuration distribution, $\mathcal{D}\{(C(x_1), p_1), .., (C(x_k), p_k)\}$, for $1 \leq i \leq k$. We define $\Delta(x_i) = 1 - f(x_i)$, where $f(x_i)$ is defined in Equation (1).

A.1 The Reduction of Mediators for $c > 1$

In this section, we consider the case that $c > 1$.

Lemma 4 *For any mediator over k configurations, and for $1 \leq i \leq k$, $\Delta(x_i) > 0$ iff ($\frac{c-1}{s_1} < x_i < \frac{1}{s_2} + \frac{c}{s_1}$ and $f(1) \geq 1$) or ($\frac{c-1}{s_1} < x_i \leq 1$ and $f(1) < 1$); and $\Delta(x_i) < 0$ iff $0 \leq x_i < \frac{c-1}{s_1}$ or ($\frac{1}{s_2} + \frac{c}{s_1} < x_i \leq 1$ and $f(1) > 1$).*

Proof. Recall that $\Delta(x_i) = 1 - f(x_i)$. Then by Equation (1), we have

$$\Delta(x_i) = \begin{cases} \Delta_1(x_i) & 0 \leq x_i \leq \frac{c}{s_1}, \\ \Delta_2(x_i) & \frac{c}{s_1} \leq x_i \leq 1. \end{cases}$$

where $\Delta_1(x_i) = 1 - (c - s_1 x_i)$ and $\Delta_2(x_i) = 1 - s_2(x_i - \frac{c}{s_1})$. Now we make a case analysis:

Case 1: $0 \leq x_i \leq \frac{c}{s_1}$: $\Delta_1(x_i) < 0 \iff 0 \leq x_i < \frac{c-1}{s_1}$; and $\Delta_1(x_i) > 0 \iff \frac{c-1}{s_1} < x_i \leq \frac{c}{s_1}$.

Case 2: $\frac{c}{s_1} \leq x_i \leq 1$: $\Delta_2(x_i) > 0 \iff (\frac{c}{s_1} \leq x_i < \frac{1}{s_2} + \frac{c}{s_1}$ and $f(1) \geq 1$) or ($\frac{c}{s_1} \leq x_i \leq 1$ and $f(1) < 1$); and $\Delta_2(x_i) < 0 \iff (\frac{1}{s_2} + \frac{c}{s_1} < x_i \leq 1$ and $f(1) > 1$). □

Lemma 5 *$\mathcal{D}\{(C(x_1), p_1), .., (C(x_k), p_k)\}$ is a correlated equilibrium iff*

$$\sum_{i=1}^{k} p_i x_i \Delta(x_i) \geq 0 \tag{2}$$

and

$$\sum_{i=1}^{k} p_i (1 - x_i) \Delta(x_i) \leq 0. \tag{3}$$

Proof. Recall that $\mathcal{E}_G^i$ is the event that the mediator advises player i to go, C_G^i is the cost for player i to go, $\mathcal{E}_S^i$ is the event that the mediator advises player i to stay, and C_S^i is the cost for player i to stay. Also we will omit the index i since the players are symmetric.

By definition, $\mathcal{D}\{(C(x_1), p_1), .., (C(x_k), p_k)\}$ is a correlated equilibrium iff

$$\mathbf{E}\left[C_S | \mathcal{E}_G\right] \geq \mathbf{E}\left[C_G | \mathcal{E}_G\right],$$
$$\mathbf{E}\left[C_G | \mathcal{E}_S\right] \geq \mathbf{E}\left[C_S | \mathcal{E}_S\right].$$

Note that:

$$\mathbf{E}\left[C_S | \mathcal{E}_G\right] = 1,$$

$$\mathbf{E}\left[C_G | \mathcal{E}_G\right] = \frac{\sum_{i=1}^k p_i f(x_i) x_i}{\sum_{i=1}^k p_i x_i},$$

$$\mathbf{E}\left[C_G | \mathcal{E}_S\right] = \frac{\sum_{i=1}^k p_i f(x_i)(1 - x_i)}{\sum_{i=1}^k p_i (1 - x_i)}$$

and

$$E(C_S | \mathcal{E}_S) = 1.$$

Therefore, $\mathcal{D}\{(C(x_1), p_1), .., (C(x_k), p_k)\}$ is a correlated equilibrium iff

$$\frac{\sum_{i=1}^k p_i f(x_i) x_i}{\sum_{i=1}^k p_i x_i} \leq 1 \tag{4}$$

and

$$\frac{\sum_{i=1}^k p_i f(x_i)(1 - x_i)}{\sum_{i=1}^k p_i (1 - x_i)} \geq 1. \tag{5}$$

By rearranging Inequalities (4) and (5), we have

$$\sum_{i=1}^k p_i x_i (1 - f(x_i)) \geq 0$$

and

$$\sum_{i=1}^k p_i (1 - x_i)(1 - f(x_i)) \leq 0.$$

$\square$

Lemma 6 *The expected social cost of* $\mathcal{D}\{(C(x_1), p_1), .., (C(x_k), p_k)\}$ *is*

$$(1 - \sum_{i=1}^{k} p_i x_i \Delta(x_i))n.$$

Proof. Let $Cost(C(x_i))$ be the cost of configuration $C(x_i)$ in $\mathcal{D}\{(C(x_1), p_1), .., (C(x_k), p_k)\}$, for $1 \leq i \leq k$. We know that the expected social cost of $\mathcal{D}\{(C(x_1), p_1), .., (C(x_k), p_k)\}$ is

$$\sum_{i=1}^{k} p_i Cost(C(x_i)).$$

We have $Cost(C(x_i)) = (x_i f(x_i) + (1 - x_i))n$, and since $\Delta(x_i) = 1 - f(x_i)$, it follows that $Cost(C(x_i)) = (1 - x_i \Delta(x_i))n$. Therefore, the expected social cost is

$$\sum_{i=1}^{k} p_i (1 - x_i \Delta(x_i))n,$$

or equivalently,

$$(\sum_{i=1}^{k} p_i - \sum_{i=1}^{k} p_i x_i \Delta(x_i))n.$$

Finally, we note that $\sum_{i=1}^{k} p_i = 1$. □

Lemma 7 *For any optimal mediator over $k \geq 2$ configurations, $\Delta(x_i) \neq 0$ for all $1 \leq i \leq k$, and $\Delta(x_u) > 0$ and $\Delta(x_v) < 0$ for some $1 \leq u, v \leq k$.*

Proof. First we show that for any optimal mediator over $k \geq 2$ configurations, $\Delta(x_i)$ is non-zero for all $1 \leq i \leq k$. Assume by way of contradiction that there is an optimal mediator M_k that uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_k), p_k)\}$, and there is some $1 \leq j \leq k$ such that $\Delta(x_j) = 0$. Recall that $0 < p_j < 1$. Now let $\mathcal{D}\{(C(x_1), \frac{p_1}{1-p_j}), .., (C(x_{j-1}), \frac{p_{j-1}}{1-p_j}), (C(x_{j+1}), \frac{p_{j+1}}{1-p_j}), .., (C(x_k), \frac{p_k}{1-p_j})\}$ be a configuration distribution over $k - 1$ configurations.

Since M_k is a mediator and $\Delta(x_j) = 0$, Constraints (2) and (3) of Lemma 5 imply that

$$\sum_{1 \leq i \leq k, i \neq j} p_i x_i \Delta(x_i) \geq 0$$

and

$$\sum_{1 \leq i \leq k, i \neq j} p_i (1 - x_i) \Delta(x_i) \leq 0.$$

Now if we multiply both sides of these two constraints by $\frac{1}{1-p_j}$, we have

$$\sum_{1 \leq i \leq k, i \neq j} \frac{p_i}{1 - p_j} x_i \Delta(x_i) \geq 0$$

and

$$\sum_{1 \leq i \leq k, i \neq j} \frac{p_i}{1 - p_j} (1 - x_i) \Delta(x_i) \leq 0.$$

By Lemma 5, $\mathcal{D}\{(C(x_1), \frac{p_1}{1-p_j}), .., (C(x_{j-1}), \frac{p_{j-1}}{1-p_j}), (C(x_{j+1}), \frac{p_{j+1}}{1-p_j}), .., (C(x_k), \frac{p_k}{1-p_j})\}$
is a correlated equilibrium. Let M_{k-1} be a mediator that uses this correlated
equilibrium. By Lemma 6, the expected social cost of M_{k-1} is

$$(1 - \frac{1}{1 - p_j} \sum_{1 \leq i \leq k, i \neq j} p_i x_i \Delta(x_i))n,$$

and since $\Delta(x_j) = 0$, the expected social cost of M_k is

$$(1 - \sum_{1 \leq i \leq k, i \neq j} p_i x_i \Delta(x_i))n.$$

We know that $0 < p_j < 1$ implies $\frac{1}{1-p_j} > 1$. Therefore, the expected social cost
M_{k-1} is less than the expected social cost of M_k. This contradicts the fact that
M_k is optimal.
Recall that $0 < p_i < 1$ and $0 \leq x_i \leq 1$ for all $1 \leq i \leq k$. By Lemma 5, Constraint
(2) implies that there exists u such that $\Delta(x_u) > 0$ for $1 \leq u \leq k$. Similarly,
Constraint (3) implies that there exists v such that $\Delta(x_v) < 0$ for $1 \leq v \leq k$.   □

Lemma 8 *Any optimal mediator that uses* $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j), p_j), ..,$
$(C(x_k), p_k)\}$, *where* $k \geq 2$, *has*

$$\sum_{1 \leq i \leq k, i \neq j} p_i \Delta(x_i)(x_i - x_j) \geq 0, 1 \leq j \leq k.$$

Proof. Let M_k be an optimal mediator that uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j), p_j), ..,$
$(C(x_k), p_k)\}$. We know by Lemma 7 that any configuration, $C(x_j)$, has either
$\Delta(x_j) < 0$ or $\Delta(x_j) > 0$, for $1 \leq j \leq k$. Now fix any $1 \leq j \leq k$, and do a case
analysis for $\Delta(x_j)$.
Case 1: If $\Delta(x_j) < 0$, then by repeated application of Lemma 5 we have

$$\frac{\sum_{1 \leq i \leq k, i \neq j} p_i \Delta(x_i)(1 - x_i)}{(1 - x_j)|\Delta(x_j)|} \leq p_j \leq \frac{\sum_{1 \leq i \leq k, i \neq j} p_i \Delta(x_i)x_i}{x_j|\Delta(x_j)|} \qquad (6)$$

Removing p_j from Inequality (6) and rearranging, we get

$$x_j|\Delta(x_j)| \sum_{1 \leq i \leq k, i \neq j} p_i \Delta(x_i)(1 - x_i) \leq (1 - x_j)|\Delta(x_j)| \sum_{1 \leq i \leq k, i \neq j} p_i \Delta(x_i)x_i.$$

By canceling the common terms, we have

$$\sum_{1 \leq i \leq k, i \neq j} p_i \Delta(x_i)x_j \leq \sum_{1 \leq i \leq k, i \neq j} p_i \Delta(x_i)x_i.$$

Case 2: If $\Delta(x_j) > 0$, then similarly by repeated application of Lemma 5 we have

$$\frac{-\sum_{1 \leq i \leq k, i \neq j} p_i \Delta(x_i) x_i}{x_j \Delta(x_j)} \leq p_j \leq \frac{-\sum_{1 \leq i \leq k, i \neq j} p_i \Delta(x_i)(1 - x_i)}{(1 - x_j)\Delta(x_j)} \qquad (7)$$

Removing p_j from Inequality (7) and rearranging, we get

$$x_j \Delta(x_j) \sum_{1 \leq i \leq k, i \neq j} p_i \Delta(x_i)(1 - x_i) \leq (1 - x_j)\Delta(x_j) \sum_{1 \leq i \leq k, i \neq j} p_i \Delta(x_i) x_i.$$

By canceling the common terms, we have

$$\sum_{1 \leq i \leq k, i \neq j} p_i \Delta(x_i) x_j \leq \sum_{1 \leq i \leq k, i \neq j} p_i \Delta(x_i) x_i.$$

Since j is any value between 1 and k, this implies the statement of the lemma for every such j. $\qquad \square$

Lemma 9 *Consider any mediator M_k that uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j), p_j), .., (C(x_k), p_k)\}$, where $0 < x_j < \frac{c-1}{s_1}$. Then there exists a mediator M_k' of less expected social cost, which uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j'), p_j), .., (C(x_k), p_k)\}$, where $x_j' = 0$.*

Proof. Let M_k be a mediator that uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j), p_j), .., (C(x_k), p_k)\}$, where $0 < x_j < \frac{c-1}{s_1}$. By Lemma 5, we have

$$p_j x_j \Delta(x_j) + \sum_{1 \leq i \leq k, i \neq j} p_i x_i \Delta(x_i) \geq 0 \qquad (8)$$

and

$$p_j (1 - x_j) \Delta(x_j) + \sum_{1 \leq i \leq k, i \neq j} p_i (1 - x_i) \Delta(x_i) \leq 0. \qquad (9)$$

Since $0 < x_j < \frac{c-1}{s_1}$, by Lemma 4, $\Delta(x_j) < 0$. Now let $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j'), p_j), .., (C(x_k), p_k)\}$ be a configuration distribution that has $x_j' = 0$. Thus, we have $p_j x_j' \Delta(x_j') = 0$ and $p_j x_j \Delta(x_j) < 0$. By Inequality (8), we have

$$p_j x_j' \Delta(x_j') + \sum_{1 \leq i \leq k, i \neq j} p_i x_i \Delta(x_i) > 0. \qquad (10)$$

We know that $\Delta(x_j') < \Delta(x_j) < 0$ and $(1 - x_j') > (1 - x_j) > 0$, so we have $(1 - x_j')\Delta(x_j') < (1 - x_j)\Delta(x_j)$. By Inequality (9), we get

$$p_j \Delta(x_j') + \sum_{1 \leq i \leq k, i \neq j} p_i (1 - x_i) \Delta(x_i) < 0. \qquad (11)$$

Now by Lemma 5 and Inequalities (10) and (11), $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j'), p_j) , .., (C(x_k), p_k)\}$ is a correlated equilibrium. Let M_k' be a mediator that uses this

correlated equilibrium. By Lemma 6, and since $x'_j = 0$, the expected social cost of M'_k is

$$(1 - \sum_{1 \le i \le k, i \ne j} p_i x_i \Delta(x_i))n.$$

Moreover, by Lemma 6, the expected social cost of M_k is

$$((1 - \sum_{1 \le i \le k, i \ne j} p_i x_i \Delta(x_i)) - p_j x_j \Delta(x_j))n.$$

Since $\Delta(x_j) < 0$ and $x_j > 0$, the expected social cost of M'_k is less than the expected social cost of M_k. $\qquad\square$

Lemma 10 *For $f(1) \ge 1$, consider any mediator M_k that uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j), p_j), .., (C(x_k), p_k)\}$, where $\frac{c-1}{s_1} < x_j < \frac{c}{s_1}$. Then there exists a mediator M'_k of less expected social cost, which uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x'_j), p_j), .., (C(x_k), p_k)\}$, where $\frac{c}{s_1} < x'_j < \frac{c}{s_1} + \frac{1}{s_2}$ and $f(x'_j) = f(x_j)$.*

Proof. Let M_k be a mediator that uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j), p_j), .., (C(x_k), p_k)\}$, where $\frac{c-1}{s_1} < x_j < \frac{c}{s_1}$. By Lemma 5, we have

$$p_j x_j \Delta(x_j) + \sum_{1 \le i \le k, i \ne j} p_i x_i \Delta(x_i) \ge 0 \qquad (12)$$

and

$$p_j(1 - x_j)\Delta(x_j) + \sum_{1 \le i \le k, i \ne j} p_i(1 - x_i)\Delta(x_i) \le 0. \qquad (13)$$

Recall that $\frac{c-1}{s_1} < x_j < \frac{c}{s_1}$ and $f(1) \ge 1$. Then $\exists x'_j : \frac{c}{s_1} < x'_j < \frac{c}{s_1} + \frac{1}{s_2}$ and $f(x'_j) = f(x_j)$. Now let $\mathcal{D}\{(C(x_1), p_1), .., (C(x'_j), p_j), .., (C(x_k), p_k)\}$ be a configuration distribution. Since $f(x'_j) = f(x_j)$, $\Delta(x'_j) = \Delta(x_j)$. We know that $x'_j > x_j$, then $x'_j \Delta(x'_j) > x_j \Delta(x_i)$. By Inequality (12), we obtain

$$p_j x'_j \Delta(x'_j) + \sum_{1 \le i \le k, i \ne j} p_i x_i \Delta(x_i) > 0.$$

Since $(1 - x'_j) < (1 - x_j)$, we have $(1 - x'_j)\Delta(x'_j) < (1 - x_j)\Delta(x_j)$. By Inequality (13), we get

$$p_j(1 - x'_j)\Delta(x'_j) + \sum_{1 \le i \le k, i \ne j} p_i(1 - x_i)\Delta(x_i) < 0.$$

Now by Lemma 5, $\mathcal{D}\{(C(x_1), p_1), .., (C(x'_j), p_j), .., (C(x_k), p_k)\}$ is a correlated equilibrium. Let M'_k be a mediator that uses this correlated equilibrium. By Lemma 6, the expected social cost of M'_k is

$$((1 - \sum_{1 \le i \le k, i \ne j} p_i x_i \Delta(x_i)) - p_j x'_j \Delta(x'_j))n,$$

and the expected social cost of M_k is

$$((1 - \sum_{1 \leq i \leq k, i \neq j} p_i x_i \Delta(x_i)) - p_j x_j \Delta(x_j))n.$$

Since $p_j x_j' \Delta(x_j') > p_j x_j \Delta(x_i)$, the expected social cost of M_k' is less than the expected social cost of M_k. $\qquad\square$

Lemma 11 *For $f(1) < 1$, consider any mediator M_k that uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j), p_j), .., (C(x_k), p_k)\}$, where $\frac{c-1}{s_1} < x_j < \frac{c}{s_1}$ and $f(x_j) \leq f(1)$. Then there exists a mediator M_k' of less expected social cost, which uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j'), p_j), .., (C(x_k), p_k)\}$, where $\frac{c}{s_1} < x_j' \leq 1$ and $f(x_j') = f(x_j)$.*

Proof. We know that for $\frac{c-1}{s_1} < x_j < \frac{c}{s_1}$ and $f(x_j) \leq f(1) < 1$, $\exists x_j' : \frac{c}{s_1} < x_j' \leq 1$ and $f(x_j') = f(x_j)$. In a manner similar to the proof of Lemma 10, we prove this Lemma. $\qquad\square$

Lemma 12 *For $f(1) < 1$, consider any mediator M_k that uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j), p_j), .., (C(x_k), p_k)\}$, where $\frac{c-1}{s_1} < x_j < \frac{c}{s_1}$ and $f(x_j) > f(1)$. Then there exists a mediator M_k' of less expected social cost, which uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j'), p_j), .., (C(x_k), p_k)\}$, where $x_j' = 1$.*

Proof. Let M_k be a mediator that uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j), p_j), .., (C(x_k), p_k)\}$, where $\frac{c-1}{s_1} < x_j < \frac{c}{s_1}$. By Lemma 5, we have

$$p_j x_j \Delta(x_j) + \sum_{1 \leq i \leq k, i \neq j} p_i x_i \Delta(x_i) \geq 0 \qquad (14)$$

and

$$p_j(1 - x_j)\Delta(x_j) + \sum_{1 \leq i \leq k, i \neq j} p_i(1 - x_i)\Delta(x_i) \leq 0. \qquad (15)$$

Now let $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j'), p_j), .., (C(x_k), p_k)\}$ be a configuration distribution, where $x_j' = 1$. For $f(x_j) > f(x_j')$ and $f(x_j') < 1$, $\Delta(x_j') > \Delta(x_j)$. We know that $x_j' > x_j$, then $x_j'\Delta(x_j') > x_j\Delta(x_i)$. By Inequality (14), we obtain

$$p_j x_j' \Delta(x_j') + \sum_{1 \leq i \leq k, i \neq j} p_i x_i \Delta(x_i) > 0.$$

Since $(1 - x_j') = 0$, $(1 - x_j) > 0$ and $\Delta(x_j) > 0$, $(1 - x_j')\Delta(x_j') < (1 - x_j)\Delta(x_j)$. By Inequality (15), we get

$$p_j(1 - x_j')\Delta(x_j') + \sum_{1 \leq i \leq k, i \neq j} p_i(1 - x_i)\Delta(x_i) < 0.$$

Now by Lemma 5, $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j'), p_j), .., (C(x_k), p_k)\}$ is a correlated equilibrium. Let M_k' be a mediator that uses this correlated equilibrium. By Lemma 6, the expected social cost of M_k' is

$$((1 - \sum_{1 \leq i \leq k, i \neq j} p_i x_i \Delta(x_i)) - p_j x_j' \Delta(x_j'))n,$$

and the expected social cost of M_k is

$$((1 - \sum_{1 \leq i \leq k, i \neq j} p_i x_i \Delta(x_i)) - p_j x_j \Delta(x_j))n.$$

Since $p_j x_j' \Delta(x_j') > p_j x_j \Delta(x_i)$, the expected social cost of M_k' is less than the expected social cost of M_k. $\square$

Lemma 13. *For any (c, s_1, s_2)-El Farol game, any optimal mediator that uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_k), p_k)\}$ has exactly one configuration that has no players advised to go, and any other configuration has at least a $\frac{c}{s_1}$-fraction of players advised to go.*

Proof. Let M_k be an optimal mediator over $k \geq 2$ configurations that uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j), p_j), .., (C(x_k), p_k)\}$. First we prove that M_k must have a configuration where no players advised to go. We know by Lemma 7 that there exists some $1 \leq j \leq k$ where $\Delta(x_j) < 0$. By Lemma 4, $\Delta(x_j) < 0$ iff $(x_j \in (1/s_2 + c/s_1, 1]$ *and* $f(1) > 1)$ or $x_j \in [0, \frac{c-1}{s_1})$. Now we do a case analysis for x_j.

Case 1: $x_j \in (1/s_2 + c/s_1, 1]$ *and* $f(1) > 1$. Assume by way of contradiction that M_k has no configuration that has less than a $\frac{c-1}{s_1}$-fraction of players advised to go. Let x_q be the smallest fraction that is $x_q > 1/s_2 + c/s_1$, where $1 \leq q \leq k$. By Lemmas 4 and 7, for $1 \leq r \leq k$,

$$\Delta(x_r) = \begin{cases} > 0 & \text{if } x_r < x_q, \\ < 0 & \text{otherwise.} \end{cases}$$

Note that by the definition of the configuration distribution, if $x_r = x_q$ then $r = q$. Therefore, we have

$$\sum_{1 \leq r \leq k, r \neq q} p_r \Delta(x_r)(x_r - x_q) < 0. \tag{16}$$

By Lemma 8, Inequality (16) contradicts that M_k is an optimal mediator. Thus, M_k must have a configuration, $C(x)$, where $x < \frac{c-1}{s_1}$, and the rest of the argument is as in Case 2.

Case 2: $x_j \in [0, \frac{c-1}{s_1})$. By Lemma 9, and since M_k is an optimal mediator, $x_j = 0$.

By the definition of the configuration distribution, M_k has no two configurations that have the same fraction of players that are advised to go. So M_k has exactly one configuration, over all the k configurations, that has no players advised to go.

We know that $\Delta(\frac{c-1}{s_1}) = 0$. By Lemma 7, there is no optimal mediator that has a configuration $C(\frac{c-1}{s_1})$. Now since M_k is an optimal mediator, by Lemmas 9, 10, 11 and 12, M_k has no configuration in which an x-fraction of players is advised to go, where $x \in (0, \frac{c}{s_1})$. $\square$

Lemma 14 *For any* $\mathcal{D}\{(C(x_1), p_1), .., (C(x_i), p_i), .., (C(x_j), p_j), .., (C(x_k), p_k)\}$, *and for any arbitrary* x_i *and* x_j *such that* $x_j > x_i \geq \frac{c}{s_1}$, *there exists* $\mathcal{D}\{(C(x_1), p_1),$ $.., (C(x_{i-1}), p_{i-1}), (C(x_{i+1}), p_{i+1}), .., (C(x_j'), p_i + p_j), .., (C(x_k), p_k)\}$, *where* $x_j' = \frac{p_i}{p_i + p_j} x_i + \frac{p_j}{p_i + p_j} x_j$. *Moreover,*

$$1)\ (p_i + p_j) x_j' \Delta(x_j') > p_i x_i \Delta(x_i) + p_j x_j \Delta(x_j).$$

$$2)\ (p_i + p_j)(1 - x_j') \Delta(x_j') < p_i(1 - x_i) \Delta(x_i) + p_j(1 - x_j) \Delta(x_j).$$

Proof. Let $\mathcal{D}\{(C(x_1), p_1), .., (C(x_i), p_i), .., (C(x_j), p_j), .., (C(x_k), p_k)\}$ be a configuration distribution that has $x_j > x_i \geq \frac{c}{s_1}$.

Also let $\mathcal{D}\{(C(x_1), p_1), .., (C(x_{i-1}), p_{i-1}), (C(x_{i+1}), p_{i+1}), .., (C(x_j'), p_i + p_j), .., (C(x_k), p_k)\}$ be a configuration distribution that has $x_j' = \frac{p_i}{p_i + p_j} x_i + \frac{p_j}{p_i + p_j} x_j$. We know that $0 < p_i, p_j < 1$ and $x_j > x_i$. Thus $x_i < x_j' < x_j$. Assume by way of contradiction that

$$(p_i + p_j) x_j' \Delta(x_j') \leq p_i x_i \Delta(x_i) + p_j x_j \Delta(x_j),$$

or equivalently,

$$x_j' \Delta(x_j') \leq \frac{p_i}{p_i + p_j} x_i \Delta(x_i) + \frac{p_j}{p_i + p_j} x_j \Delta(x_j).$$

Let $p = \frac{p_i}{p_i + p_j}$, so $1 - p = \frac{p_j}{p_i + p_j}$. Then we have

$$x_j' \Delta(x_j') \leq p x_i \Delta(x_i) + (1 - p) x_j \Delta(x_j).$$

Recall that for $\frac{c}{s_1} \leq x \leq 1$, $\Delta(x) = 1 - s_2(x - \frac{c}{s_1})$. Since $\frac{c}{s_1} \leq x_i, x_j, x_j' \leq 1$, we get

$$x_j'(1 - s_2(x_j' - \frac{c}{s_1})) \leq p x_i(1 - s_2(x_i - \frac{c}{s_1})) + (1 - p) x_j(1 - s_2(x_j - \frac{c}{s_1})).$$

Since $x_j' = p x_i + (1 - p) x_j$, we have

$$x_j'(-s_2(x_j' - \frac{c}{s_1})) \leq p x_i(-s_2(x_i - \frac{c}{s_1})) + (1 - p) x_j(-s_2(x_j - \frac{c}{s_1})).$$

We know that $s_2 > 0$, and hence dividing by $-s_2$, we get

$$x_j'(x_j' - \frac{c}{s_1}) \geq p x_i(x_i - \frac{c}{s_1}) + (1 - p) x_j(x_j - \frac{c}{s_1}).$$

Since $-\frac{c}{s_1} x_j' = -\frac{c}{s_1}(p x_i + (1 - p) x_j)$, we have

$$x_j'^2 \geq p x_i^2 + (1 - p) x_j^2.$$

Substituting x_j' by $p x_i + (1 - p) x_j$, we get

$$p^2 x_i^2 + 2p(1 - p) x_i x_j + (1 - p)^2 x_j^2 \geq p x_i^2 + (1 - p) x_j^2.$$

By rearranging, we have

$$p(1-p)(x_j^2 - 2x_ix_j + x_i^2) \leq 0.$$

Now since $0 < p < 1$, we can divide by $p(1-p)$, and we get

$$(x_j - x_i)^2 \leq 0,$$

which contradicts since $x_j \neq x_i$. This proves that

$$(p_i + p_j)x_j'\Delta(x_j') > p_ix_i\Delta(x_i) + p_jx_j\Delta(x_j). \tag{17}$$

Now we prove that $(p_i + p_j)(1 - x_j')\Delta(x_j') < p_i(1-x_i)\Delta(x_i) + p_j(1-x_j)\Delta(x_j)$. To do so, we first show that $(p_i + p_j)\Delta(x_j') = p_i\Delta(x_i) + p_j\Delta(x_j)$.

We know that

$$x_j' = \frac{p_i}{p_i + p_j}x_i + \frac{p_j}{p_i + p_j}x_j$$

$$\Longleftrightarrow (p_i + p_j)x_j' = p_ix_i + p_jx_j$$

$$\Longleftrightarrow (p_i + p_j)(x_j' - \frac{c}{s_1}) = p_i(x_i - \frac{c}{s_1}) + p_j(x_j - \frac{c}{s_1})$$

$$\Longleftrightarrow (p_i + p_j)s_2(x_j' - \frac{c}{s_1}) = p_is_2(x_i - \frac{c}{s_1}) + p_js_2(x_j - \frac{c}{s_1})$$

$$\Longleftrightarrow (p_i + p_j)(1 - s_2(x_j' - \frac{c}{s_1})) = p_i(1 - s_2(x_i - \frac{c}{s_1})) + p_j(1 - s_2(x_j - \frac{c}{s_1}))$$

Recall that $\Delta(x) = 1 - s_2(x - \frac{c}{s_1})$ when $\frac{c}{s_1} \leq x \leq 1$. Since $\frac{c}{s_1} \leq x_i, x_j, x_j' \leq 1$, we get

$$(p_i + p_j)\Delta(x_j') = p_i\Delta(x_i) + p_j\Delta(x_j). \tag{18}$$

By subtracting (17) from (18), we obtain

$$(p_i + p_j)(1 - x_j')\Delta(x_j') < p_i(1-x_i)\Delta(x_i) + p_j(1-x_j)\Delta(x_j).$$

$\square$

Lemma 15. *For any (c, s_1, s_2)-El Farol game, any optimal mediator that uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_k), p_k)\}$ has at most one configuration that has at least a $\frac{c}{s_1}$-fraction of players advised to go, and any other configuration has less than a $\frac{c}{s_1}$-fraction of players advised to go.*

Proof. Assume by way of contradiction that there is an optimal mediator M_k that uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_i), p_i), .., (C(x_j), p_j), .., (C(x_k), p_k)\}$, where $x_j > x_i \geq \frac{c}{s_1}$. Let $\mathcal{D}\{(C(x_1), p_1), .., (C(x_{i-1}), p_{i-1}), (C(x_{i+1}), p_{i+1}), .., (C(x_j'), p_i + p_j), .., (C(x_k), p_k)\}$ be a configuration distribution that has $x_j' = \frac{p_i}{p_i + p_j}x_i + \frac{p_j}{p_i + p_j}x_j$. Since M_k is a mediator, by Lemma 5, we have

$$p_ix_i\Delta(x_i) + p_jx_j\Delta(x_j) + \sum_{1 \leq r \leq k, r \neq i, r \neq j} p_rx_r\Delta(x_r) \geq 0 \tag{19}$$

and

$$p_i(1 - x_i)\Delta(x_i) + p_j(1 - x_j)\Delta(x_j) + \sum_{1 \leq r \leq k, r \neq i, r \neq j} p_r(1 - x_r)\Delta(x_r) \leq 0. \quad (20)$$

By Lemma 14, we have

$$(p_i + p_j)x_j'\Delta(x_j') > p_i x_i \Delta(x_i) + p_j x_j \Delta(x_j) \quad (21)$$

and

$$(p_i + p_j)(1 - x_j')\Delta(x_j') < p_i(1 - x_i)\Delta(x_i) + p_j(1 - x_j)\Delta(x_j). \quad (22)$$

By Inequalities (19) and (21), we get

$$(p_i + p_j)x_j'\Delta(x_j') + \sum_{1 \leq r \leq k, r \neq i, r \neq j} p_r x_r \Delta(x_r) > 0. \quad (23)$$

Similarly, by Inequalities (20) and (22), we obtain

$$(p_i + p_j)(1 - x_j')\Delta(x_j') + \sum_{1 \leq r \leq k, r \neq i, r \neq j} p_r(1 - x_r)\Delta(x_r) < 0. \quad (24)$$

By Lemma 5 and Inequalities (23) and (24), $\mathcal{D}\{(C(x_1), p_1), .., (C(x_{i-1}), p_{i-1}), (C(x_{i+1}), p_{i+1}), .., (C(x_j'), p_i + p_j), .., (C(x_k), p_k)\}$ is a correlated equilibrium. Let M_{k-1} be a mediator that uses this correlated equilibrium. By Lemma 6, the expected social cost of M_k is

$$((1 - \sum_{1 \leq r \leq k, r \neq i, r \neq j} p_r x_r \Delta(x_r)) - p_i x_i \Delta(x_i) - p_j x_j \Delta(x_j))n,$$

and the expected social cost of M_{k-1} is

$$((1 - \sum_{1 \leq r \leq k, r \neq i, r \neq j} p_r x_r \Delta(x_r)) - (p_i + p_j)x_j'\Delta(x_j'))n.$$

Since $(p_i + p_j)x_j'\Delta(x_j') > p_i x_i \Delta(x_i) + p_j x_j \Delta(x_j)$, the expected social cost of M_{k-1} is less than the expected social cost of M_k. This contradicts that M_k is an optimal mediator. $\square$

Lemma 16. *For any (c, s_1, s_2)-El Farol game, there exists an optimal mediator that uses $\mathcal{D}\{(C(0), p), (C(x), 1 - p)\}$, where $0 < p < 1$; and $\frac{c}{s_1} \leq x < \frac{1}{s_2} + \frac{c}{s_1}$ if $f(1) \geq 1$, otherwise $\frac{c}{s_1} \leq x \leq 1$.*

Proof. By the definition of the configuration distribution, a mediator has at least two configurations. By Lemmas 13 and 15, there exists an optimal mediator that has exactly two configurations. The first configuration has no players advised to go, and the second configuration has an x-fraction of players advised to go, where $x \geq \frac{c}{s_1}$. Since the first configuration has *zero* players advised to go, by Lemma 4, $\Delta(0) < 0$. By Lemma 7, we must have $\Delta(x) > 0$. We know that $x \geq \frac{c}{s_1}$. By Lemma 4, if $f(1) \geq 1$, then $\frac{c}{s_1} \leq x < \frac{1}{s_2} + \frac{c}{s_1}$; otherwise, $\frac{c}{s_1} \leq x \leq 1$. $\square$

A.2   The Reduction of Mediators for $c \leq 1$

Now we consider the case that $c \leq 1$ in the following lemma.

Lemma 17. *For any (c, s_1, s_2)-El Farol game , if $c \leq 1$, then $\mathrm{MV} = 1$.*

Proof. In a manner similar to Lemma 7, any optimal mediator over $k \geq 2$ does not have a configuration $C(x)$ with $\Delta(x) = 0$.

Also in a manner similar to Lemmas 10, 11 and 12, for any mediator M_k that uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j), p_j), .., (C(x_k), p_k)\}$, where $0 \leq x_j < \frac{c}{s_1}$, there exists a mediator M_k' of less expected social cost, which uses $\mathcal{D}\{(C(x_1), p_1), .., (C(x_j'), p_j), .., (C(x_k), p_k)\}$, where $\frac{c}{s_1} \leq x_j' < \frac{1}{s_2} + \frac{c}{s_1}$ if $f(1) \geq 1$; otherwise, $\frac{c}{s_1} \leq x_j' \leq 1$.

Finally, in a manner similar to Lemma 15, any optimal mediator has at most one configuration, $C(x)$, where $x \geq \frac{c}{s_1}$.

Therefore, for $c \leq 1$, the best correlated equilibrium is a configuration distribution over just one configuration, which is trivially the best Nash equilibrium.

A.3   An Optimal Mediator

We have proved that for any (c, s_1, s_2)-El Farol game, if $c \leq 1$ then the best correlated equilibrium is the best Nash equilibrium; otherwise, there exists an optimal mediator that is over two configurations. Now we describe this mediator in detail.

Lemma 18. *For any (c, s_1, s_2)-El Farol game, if $c > 1$, then $\mathcal{D}\{(C(0), p), (C(x), 1-p)\}$ is the best correlated equilibrium, where $\lambda(c, s_1, s_2) = c(\frac{1}{s_1} + \frac{1}{s_2}) - \sqrt{\frac{c(\frac{1}{s_1} + \frac{1}{s_2})(c-1)}{s_2}}$,*

$$x = \begin{cases} \lambda(c, s_1, s_2) & \text{if } \frac{c}{s_1} \leq \lambda(c, s_1, s_2) < 1, \\ \frac{c}{s_1} & \text{if } \lambda(c, s_1, s_2) < \frac{c}{s_1}, \\ 1 & \text{otherwise.} \end{cases}$$

and $p = \frac{(1-x)(1-f(x))}{(1-x)(1-f(x))+c-1}$. Moreover, the expected social cost is

$$(p + (1-p)(xf(x) + (1-x)))n.$$

Proof. By Lemma 16, there exists an optimal mediator M_2 that uses $\mathcal{D}\{(C(0), p), (C(x), 1-p)\}$, where $\frac{c}{s_1} \leq x < \frac{1}{s_2} + \frac{c}{s_1}$ if $f(1) \geq 1$; otherwise, $\frac{c}{s_1} \leq x \leq 1$.

Now we determine p and x so that M_2 is an optimal mediator.

First, we determine p. By Constraint (3) of Lemma 5, we have

$$p\Delta(0) + (1-p)(1-x)\Delta(x) \leq 0. \tag{25}$$

We know that $c > 1$, $\Delta(0) = 1 - c$ and $\Delta(x) > 0$. By rearranging Inequality (25), we obtain

$$p \geq \frac{(1-x)\Delta(x)}{(c-1) + (1-x)\Delta(x)}. \tag{26}$$

Recall that the cost of any configuration, $C(x_i)$, is $Cost(C(x_i)) = (1 - x_i \Delta(x_i))n$. Thus $Cost(C(0)) = n$, and $Cost(C(x)) = (1 - x\Delta(x))n$. Since $\Delta(x) > 0$, $Cost(C(x)) < n$. Thus, $Cost(C(x)) < Cost(C(0))$. We know that the social cost of M_2 is

$$pCost(C(0)) + (1 - p)Cost(C(x)). \qquad (27)$$

Since $Cost(C(x)) < Cost(C(0))$, the minimum expected social cost is when p is the smallest possible value in Inequality (26) which is $\frac{(1-x)\Delta(x)}{(c-1)+(1-x)\Delta(x)}$.

Now we determine x. By Lemma 6, the expected social cost of M_2 is

$$(1 - (1 - p)x\Delta(x))n.$$

Since $p = \frac{(1-x)\Delta(x)}{(c-1)+(1-x)\Delta(x)}$, the expected social cost is then

$$(1 - \frac{(c-1)x\Delta(x)}{(c-1)+(1-x)\Delta(x)})n.$$

As M_2 is an optimal mediator, we minimize its expected social cost with respect to x. Thus $g(x)$ is maximized with respect to x, where

$$g(x) = \frac{(c-1)x\Delta(x)}{(c-1)+(1-x)\Delta(x)}.$$

Hence, we have

$$\frac{dg(x)}{dx} = \frac{(c-1)[(c-1+(1-x)\Delta(x))(\Delta(x)-s_2x)+x\Delta(x)((1-x)s_2+\Delta(x))]}{((c-1)+(1-x)\Delta(x))^2}.$$

By rearranging and canceling common terms, we obtain

$$\frac{dg(x)}{dx} = \frac{(c-1)[(\Delta(x))^2+(c-1)\Delta(x)-(c-1)s_2x]}{((c-1)+(1-x)\Delta(x))^2}.$$

We know that $\Delta(x) > 0$, $\frac{c}{s_1} \le x < \frac{c}{s_1} + \frac{1}{s_2}$, $x \le 1$ and $c > 1$, so the denominator is always positive. By setting the numerator to zero and dividing by $c - 1$, we get

$$(\Delta(x))^2 + (c-1)\Delta(x) - (c-1)s_2x = 0 \qquad (28)$$

By solving Equation (28), we have $x = c(\frac{1}{s_1} + \frac{1}{s_2}) \pm \sqrt{\frac{c(\frac{1}{s_1}+\frac{1}{s_2})(c-1)}{s_2}}$. Now let $\lambda(c, s_1, s_2) = c(\frac{1}{s_1} + \frac{1}{s_2}) - \sqrt{\frac{c(\frac{1}{s_1}+\frac{1}{s_2})(c-1)}{s_2}}$ and $\bar{\lambda}(c, s_1, s_2) = (c(\frac{1}{s_1} + \frac{1}{s_2}) + \sqrt{\frac{c(\frac{1}{s_1}+\frac{1}{s_2})(c-1)}{s_2}})$.

Since $\bar{\lambda}(c, s_1, s_2) > (\frac{1}{s_2} + \frac{c}{s_1})$, by Lemma 16, it is out of range. Therefore, we have exactly one root $x = \lambda(c, s_1, s_2)$.

We know $\frac{dg(x)}{dx}\mid_{(x=\frac{c}{s_1})}< 0$ iff $\lambda(c,s_1,s_2) < \frac{c}{s_1}$, and $\frac{dg(x)}{dx}\mid_{(x=1)}> 0$ iff $\lambda(c,s_1,s_2) > 1$. Also we know that $\lambda(c,s_1,s_2) < \frac{c}{s_1}+\frac{1}{s_2}$; and $\frac{c}{s_1} \leq x < \frac{1}{s_2}+\frac{c}{s_1}$ if $f(1) \geq 1$, otherwise, $\frac{c}{s_1} \leq x \leq 1$. Therefore, for $\frac{c}{s_1} \leq \lambda(c,s_1,s_2) \leq 1$, the maximum of $g(x)$ is at $x = \lambda(c,s_1,s_2)$. Moreover, if $\lambda(c,s_1,s_2) < \frac{c}{s_1}$, then the maximum of $g(x)$ is at $x = \frac{c}{s_1}$; and for $\lambda(c,s_1,s_2) > 1$, the maximum is at $x = 1$.

Recall that the expected social cost of $\mathcal{D}\{(C(0),p),(C(x),1-p)\}$ is

$$pCost(C(0)) + (1-p)Cost(C(x)),$$

or equivalently,

$$(p + (1-p)(xf(x)+(1-x)))n.$$

$\square$

A.4   The Mediation Metrics

Now we compute the *Mediation Value* and the *Enforcement Value*. To obtain the *Mediation Value* and *Enforcement Value*; recall that the *Mediation Value* (MV) is the ratio of the minimum social cost over all Nash equilibria to the minimum social cost over all mediators, and the *Enforcement Value* is the ratio of the minimum social cost over all mediators to the optimal social cost.

For $c \leq 1$, by Lemma 17, $MV = 1$; and by Lemmas 1 and 2, $EV = \frac{\min(f(1),1)}{yf(y)+(1-y)}$.

For $c > 1$, by Lemmas 2 and 18, the *Mediation Value* is:

$$\frac{\min(f(1),1)}{p + (1-p)(xf(x)+(1-x))};$$

and by Lemmas 1 and 18, the *Enforcement Value* is:

$$\frac{p + (1-p)(xf(x)+(1-x))}{yf(y)+(1-y)}.$$